Modelli @ Clamfim

Lesson 1

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Euclidean space $\mathbb{R}^n$

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Euclidean Space

For any $n \in \mathbb{N}$ let $\mathbb{R}^n$ denote the n-fold Cartesian product of $\mathbb{R}$ with itself

$$\mathbb{R}^{\boldsymbol{n}} := \{(\boldsymbol{x_1}, \boldsymbol{x_2}, \dots, \boldsymbol{x_n}) : \boldsymbol{x_j} \in \mathbb{R} \text{ for } \boldsymbol{j} = \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{n}\}$$

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By Euclidean space we shall mean $\mathbb{R}^n$ together with the "Euclidean inner product" we are going to introduce. The integer n is called the dimension of $\mathbb{R}^n$, elements $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ of $\mathbb{R}^n$ are called points or vectors or ordered n-tuples, and the numbers x_j are called coordinates, or components, of $\boldsymbol{x}$

Two vectors $\boldsymbol{x}$, $\boldsymbol{y}$ are said to be equal if and only if their components are equal; i.e., $x_j = y_j$ for $j = 1$, 2, ..., n. The zero vector is the vector whose components are all zero; i.e., $\mathbf{0} := (0, 0, ..., 0)$. When $n = 2$ (respectively, $n = 3$), we usually denote the components of $\boldsymbol{x}$ by x, y (respectively, by x, y, z).

Definition. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$ be vectors and $\alpha \in \mathbb{R}$ be a scalar.

(i) The sum of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the vector

$$\boldsymbol{x} + \boldsymbol{y} := (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$$

(ii) The difference of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the vector

$$\boldsymbol{x} - \boldsymbol{y} := (x_1 - y_1, x_2 - y_2, \ldots, x_n - y_n)$$

(iii) The product of a scalar α and a vector $\boldsymbol{x}$ is the vector

$$\alpha\boldsymbol{x} = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

(iv) The (Euclidean) dot product (or scalar product or inner product) of $\boldsymbol{x}$ and $\boldsymbol{y}$ is the scalar

$$\boldsymbol{x} \cdot \boldsymbol{y} := x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

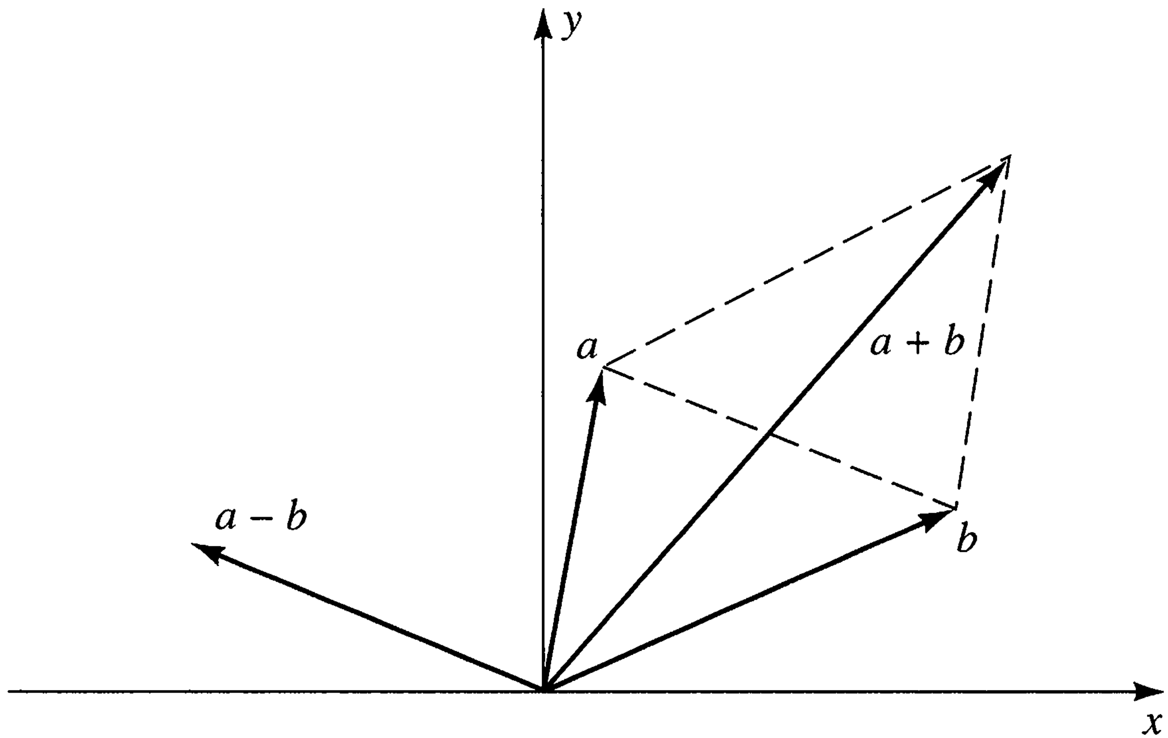

Figure 1: Vector's operations

Let $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z} \in \mathbb{R}^n$ and α, $\beta, \in \mathbb{R}$. Then

1. $\alpha\mathbf{0} = \mathbf{0}$,
2. $0\boldsymbol{x} = \mathbf{0}$,
3. $1\boldsymbol{x} = \boldsymbol{x}$,
4. $\alpha(\beta\boldsymbol{x}) = \beta(\alpha\boldsymbol{x}) = (\alpha\beta)\boldsymbol{x}$
5. $\alpha(\boldsymbol{x} \cdot \boldsymbol{y}) = (\alpha\boldsymbol{x}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot (\alpha\boldsymbol{y})$
6. $\alpha(\boldsymbol{x} + \boldsymbol{y}) = \alpha\boldsymbol{x} + \alpha\boldsymbol{y}$
7. $\mathbf{0} + \boldsymbol{x} = \boldsymbol{x}$
8. $\boldsymbol{x} - \boldsymbol{x} = \mathbf{0}$
9. $\mathbf{0} \cdot \boldsymbol{x} = 0$

10. $\boldsymbol{x + (y + z) = (x + y) + z}$

11. $\boldsymbol{x + y = y + x}$

12. $\boldsymbol{x \cdot y = y \cdot x}$

13. $\boldsymbol{x \cdot (y + z) = x \cdot y + x \cdot z}$

We define the **standard basis of $\mathbb{R}^n$** to be the collection $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, where $\boldsymbol{e}_j$ is the point in $\mathbb{R}^n$ whose j-th coordinate is 1, and all other coordinates are 0.

By definition each $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ can be written as a linear combination of the $\boldsymbol{e}_j$'s:

$$\boldsymbol{x} = \sum_{j=1}^{n} x_j \boldsymbol{e}_j = \sum_{j=1}^{n} \boldsymbol{x} \cdot \boldsymbol{e}_j \boldsymbol{e}_j$$

Definition. Let $\boldsymbol{x} \in \mathbb{R}^n$. The (Euclidean) norm (or magnitude) of $\boldsymbol{x}$ is the scalar

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Theorem: Cauchy–Schwarz inequality

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}|| \, ||\boldsymbol{y}||$$

PROOF. We consider only the non trivial situation $\boldsymbol{x}, \boldsymbol{y} \neq \mathbf{0}$.

For $t \in \mathbb{R}$ define $f(t) := ||\boldsymbol{x} - t\boldsymbol{y}||^2$

Since $0 \leq f(t) = ||\boldsymbol{x}||^2 - 2t\boldsymbol{x} \cdot \boldsymbol{y} + t^2||\boldsymbol{y}||^2$ it must be

$$\frac{\Delta}{4} = (\boldsymbol{x} \cdot \boldsymbol{y})^2 - ||\boldsymbol{x}||^2||\boldsymbol{y}||^2 \leq 0$$

and thesis follows

Norm properties. If $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{R}^n$, then

(i) $||\boldsymbol{x}|| \geq 0$ with equality only when $\boldsymbol{x} = \boldsymbol{0}$

(ii) $||\alpha\boldsymbol{x}|| = |\alpha|\,||\boldsymbol{x}||$ for all scalars α

(iii)

$$||\boldsymbol{x} + \boldsymbol{y}|| \leq ||\boldsymbol{x}|| + ||\boldsymbol{y}||$$

$$||\boldsymbol{x} - \boldsymbol{y}|| \geq ||\boldsymbol{x}|| - ||\boldsymbol{y}||$$

(iii) are said triangle inequalities

PROOF. (i) and (ii) are trivial. To get the first of (iii) observe that

$$||\boldsymbol{x}+\boldsymbol{y}||^2 = ||\boldsymbol{x}||^2 + 2\boldsymbol{x}\cdot\boldsymbol{y} + ||\boldsymbol{y}||^2$$

But from Cauchy–Schwarz inequality we infer

$$||\boldsymbol{x}||^2 + 2\boldsymbol{x}\cdot\boldsymbol{y} + ||\boldsymbol{y}||^2 \leq ||\boldsymbol{x}||^2 + 2||\boldsymbol{x}||||\boldsymbol{y}|| + ||\boldsymbol{y}||^2 = (||\boldsymbol{x}|| + ||\boldsymbol{y}||)^2$$

The second of (iii) follows analogously from

$$||\boldsymbol{x}-\boldsymbol{y}||^2 = ||\boldsymbol{x}||^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + ||\boldsymbol{y}||^2$$

and from Cauchy–Schwarz inequality